Let $\mathcal{E} \subset \mathcal{C}$ be a dg subcat. with objects $F(\mathcal{A})$.
From assumption, know $\mathcal{C} \xrightarrow[q.e.]{} P(\mathcal{E})$.

$$\mathcal{A} \xrightarrow{F} H_0(\mathcal{E}) \xleftarrow[q.\,equiv]{} \tau_{\le 0}\mathcal{E} \longrightarrow \mathcal{E}$$

because $\mathrm{Ext}^{<0} = 0$ assumption.

$\Rightarrow$ q. funct. $P(\mathcal{A}) \longrightarrow P(\mathcal{E})$

<u>Rmk 1</u>: Probably true for K = general ring (need not be field)

<u>Rmk 2</u>: Cannot apply to "periodic categories" because $\mathrm{Ext}^{<0} = 0$.

<u>Problem</u> $\quad T \xrightarrow{F} T'$

$$\mathcal{C} \overset{\exists F'}{\dashrightarrow} \mathcal{C}'$$

does a lift exist.

<u>Order</u>

<u>Part II</u>

Enhancement of Δ'ed cat.

$\mathcal{T}$ - Δ cat. $(\mathcal{B}, \varepsilon)$, $\mathcal{B}$ is dg cat (pre-Δ)

$\varepsilon: H^0(\mathcal{B}) \xrightarrow{\sim} \mathcal{T}$

<u>Defn</u>: T has unique enhancement if $(\mathcal{B}, \varepsilon)$, $(\mathcal{B}', \varepsilon')$ are 2 enhancements of T, $\exists\, F: \mathcal{B} \to \mathcal{B}'$, a quasi-functor, $H_0(F): H_0(\mathcal{B}) \xrightarrow{\sim} H_0(\mathcal{B}')$.

<u>Defn</u>: T has a strongly unique enhancement if $\exists$ $F: \mathcal{B} \longrightarrow \mathcal{B}'$ quasi-f.

$$\begin{tikzcd} H_0(\mathcal{B}) \arrow[rr, "H_0(F)"] \arrow[dr, "\varepsilon"'] & \circlearrowleft & H_0(\mathcal{B}') \arrow[dl, "\varepsilon'"] \\ & T & \end{tikzcd}$$

semi-strong uniqueness: for any object $X \in \mathcal{B}$,
$\varepsilon(X) \cong \varepsilon' \circ H_0(F)(X)$

$(\Rightarrow \underline{\mathrm{Hom}}_{\mathcal{B}}(X,X) \simeq \underline{\mathrm{Hom}}_{\mathcal{B}'}(X,X))$.

$\mathcal{A}$ - small cat

$D(\mathcal{A}) = D(\mathrm{Mod}\text{-}\mathcal{A})$

$\mathrm{Mod}\text{-}\mathcal{A}$ = dg modules

$\mathrm{Mod}\text{-}\mathcal{A}$ = modules

$L \subset D(\mathcal{A}) = H_0(\mathcal{P}(\mathcal{A})) \simeq H_0(\mathcal{SF}(\mathcal{A}))$.

localising subcat. $\quad D(\mathcal{A}) \downarrow \pi \leftarrow$ Bousfeld loc. ($\exists$ right adjoint functor)

$$H_0(\mathcal{P}(\mathcal{A})/\mathcal{L}) = D(\mathcal{A})/L.$$

Drinfeld quotient

$\mathcal{A} \hookrightarrow D(\mathcal{A})$ Yoneda.

<u>Thm</u>: Let $\mathcal{A}$, $L \subset D(\mathcal{A})$ be as above. Assume

a) $\pi(Y)$ - compact in $D(\mathcal{A})/_L$

b) $Y, Z \in \mathcal{A}$, $\mathrm{Hom}(\pi(Y), \pi(Z)[i]) = 0 \quad \forall i < 0$.

Let $\mathcal{E}$ be a dg cat. ~~$\mathcal{A}$~~ Let $F: D(\mathcal{A})/_L \longrightarrow H_0(\mathcal{E})$ be a fully faithful functor.

Then $\exists$ a quasi functor $\mathcal{F}: \mathcal{P}(\mathcal{A})/_L \longrightarrow \mathcal{E}$ s.t.

1) $H^0(\mathcal{F}): D(\mathcal{A})/_L \longrightarrow H_0(\mathcal{E})$ is also f.f. funct.

2) $H^0(\mathcal{F})(X) \cong F(X) \quad \forall X \in D(\mathcal{A})/_L$.

$\Downarrow$

Semistrong uniqueness for $D(\mathcal{A})/_L$.

<u>Thm</u>: Let $\mathcal{A}$, $L \subset D(\mathcal{A})$ be as above. L is generated by $L \cap D(\mathcal{A})^c$. Assume

$\mathrm{Hom}(\pi(Y), \pi(Z)[i]) = 0 \quad i < 0$.

Let $\mathcal{V} \subset \mathcal{P}(\mathcal{A})/_L \qquad H_0(\mathcal{V}) = \left(D(\mathcal{A})/_L\right)^c$

Let $\mathcal{E}$ be a dg cat, $N: \left(D(\mathcal{A})/_L\right)^c \longrightarrow H_0(\mathcal{E})$ f.f. functor

1) $H_0(\mathcal{N}): \left(D(\mathcal{A})/_L\right)^c \longrightarrow H_0(\mathcal{E})$ is also f.f.

$\mathcal{A} \xhookrightarrow{h} D(\mathcal{A})$.

2) $H_0(\mathcal{N})(X) \cong \mathcal{N}(X)$

3) $\exists\, \mathcal{U} \quad H_0(\mathcal{N}) \pi h \simeq N \circ \pi \circ h$

<u>Geometric applications</u>

X = quasi-proj. scheme $\qquad X \subset \bar{X} \subset \mathbb{P}^N$

$\mathcal{A} = \{ \mathcal{O}_{\bar{X}}(i) \}_{i \in \mathbb{Z}} \subset \mathrm{coh}\, \bar{X} \qquad A = \bigoplus_{i=0}^{\infty} H^0(\bar{X}, \mathcal{O}(i))$

$\mathrm{Qcoh}\, \bar{X} = \mathrm{Gr}(A)/_{\mathrm{Torsion}(A)}$ ← graded modules/A $\qquad (\mathrm{Mod}\, \mathcal{A} = \mathrm{Gr}\, A)$

$\mathrm{Qcoh}\, X = \mathrm{Gr}(A)/_{\mathrm{Gr}_I(A)} \qquad I$ = ideal

$= \mathrm{Mod}\, \mathcal{A}/_{\mathcal{N}}$

$D(\mathrm{Qcoh}\, X) = D(\mathcal{A})/_{L_{\mathcal{N}}}$ ← complexes, cohomology of which lies in $\mathcal{N}$.

<u>Thm</u>: For any quasi-~~coherent~~ projective scheme $D(\mathrm{Qcoh}\, X)$ has a unique enhancement (semi-strong).

<u>Thm</u>: For any quasi-proj. scheme $\mathrm{Perf}\, X = D(\mathrm{Qcoh}\, X)$ also has a unique enhancement.

Let $\mathcal{C}$ be a Grothendieck category (abelian cat Ab5 set of generators)

<u>Thm</u>: Let $\mathcal{C}$ be a Groth. cat. Assume it has a set of small generators s.t. they are compact objects in $D(\mathcal{C})$. Then $D(\mathcal{C})$ has a unique enhancement (semi-strong).

<u>Cor</u>: Let X be a quasi-compact, separated scheme. X has enough locally free sheaves of finite type.

Then $D(\mathrm{Qcoh}\, X)$ has a unique enhancement (semistrong).

NB. "enough loc. free sheaves of finite type" $\Leftrightarrow$ $\forall\, \mathcal{F}$ finitely presented qcoh sheaf, we have

$$\underset{\text{vector bundle}}{\mathcal{E}} \twoheadrightarrow \mathcal{F}$$

X - quasi proj. scheme

$$\mathrm{Perf}\, X \subset D^b(\mathrm{coh}\, X) \subset D(\mathrm{Qcoh}\, X)$$
$$\mathrm{Perf}\, X = D(\mathrm{Qcoh}\, X)^c$$

Thm: Let X be a quasi-projective scheme. Then $D^b(\mathrm{coh}\, X)$ has a unique enhancement.

$\mathrm{Perf}\, X$, $D^b(\mathrm{coh}\, X)$ ~~ken~~ ... strong uniqueness.

Defn: Let $\mathcal{A}$ be an abelian cat. Let $\{P_i\}_{i\in\mathbb{Z}}$ be a sequence in $\mathcal{A}$. We say $\{P_i\}_{i\in\mathbb{Z}}$ is <u>ample</u> if for any $C \in \mathcal{A}$, $\exists\, N$ s.t. for all $i < N$

a) $\exists\, P_i^{\oplus n_i} \twoheadrightarrow C$

b) $\mathrm{Ext}^j(P_i, C) = 0 \quad \forall\, j \neq 0$

c) $\mathrm{Hom}(C, P_i) = 0$.

Prop: Let $X \subset \mathbb{P}^N$ be a proj. scheme without embedding point (i.e. torsion subsheaf $T(\mathcal{O}_X) \subset \mathcal{O}_X$ with zero support $T_0(\mathcal{O}_X)$ require $T_0(\mathcal{O}_X) = 0$).

Then $\{\mathcal{O}_X(i)\}_{i\in\mathbb{Z}}$ is ample in $\mathrm{coh}\, X$.

$\underset{\text{loc. free}}{\mathrm{Loc}\, X} \subset \mathrm{coh}\, X \qquad D^b(\mathrm{Loc}\, X) = \mathrm{Perf}\, X.$

Prop: Let $\mathcal{E}$ be an exact cat with ample sequence $\{P_i\}$.

Let $F : D^b(\mathcal{E}) \xrightarrow{\sim} D^b(\mathcal{E})$, $\mathcal{P} \overset{j}{\hookrightarrow} D^b(\mathcal{E})$, $\mathcal{P} = \{P_i\}_{i\in\mathbb{Z}}$

Assume there is an iso. of functors

$\theta_{\mathcal{P}} : j \xrightarrow{\sim} F \circ j \quad \mathcal{P} \to D^b(\mathcal{E})$

$\Rightarrow \quad \mathrm{id} \cong F$ on $D^b(\mathcal{E})$.

Thm: Let $\mathcal{E}$ be an exact cat. with ample sequence $\{P_i\}_{i\in\mathbb{Z}}$.

$\mathcal{P} \subset \mathcal{E}$ ob $\mathcal{P} = \{P_i\}_{i\in\mathbb{Z}}$. Assume

$$D^b(\mathcal{E}) \xrightarrow{\sim} \left(D(\mathcal{P})/L\right)^c$$

with L = localising subcat generated by ~~compact~~ $L \cap D(\mathcal{P})^c$

Then $D^b(\mathcal{E})$ has a strongly unique enhancement.

Thm: Let X be a proj. scheme, $T_0(\mathcal{O}_X) = 0$. Then $\mathrm{Perf}\, X$, $D^b\,\mathrm{coh}\, X$ have strongly unique enhancement.

Thm (Toën): Let X, Y be quasi-compact separated schemes.

$$D_d(Qcoh(X\times Y)) \xrightarrow{\sim} R\mathcal{H}om_c(D_{dg}(Qcoh\, X), D_{dg}(Qcoh\, Y))$$

↑ cts ⟺ preserve direct sums.

$\mathcal{E} \in D(Qcoh(X\times Y))$

$P_{\mathcal{E}} : D(Qcoh\, X) \longrightarrow D(Qcoh\, Y)$

$P_{\mathcal{E}} : Rp_{2*}(p_1^*(-)\otimes \mathcal{E})$.

Cor: Let X be a quasi-pr. scheme, Y quasi-comp. & sep.
Let $K : Perf\, X \longrightarrow D(Qcoh\, Y)$ fully faithful.
Then there is an object $\mathcal{E} \in D(Qcoh(X\times Y))$ s.t.

1) $P_{\mathcal{E}}|_{Perf X} : Perf\, X \longrightarrow D(Qcoh\, Y)$ is also fully faithful and $P_{\mathcal{E}}(P') \cong K(P')$ for all $P' \in Perf\, X$.

2) if X projective, $T_0(\mathcal{O}_X)=0 \Longrightarrow P_{\mathcal{E}}|_{Perf X} \cong K$

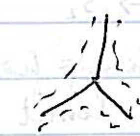
3) If K sends $Perf(X)$ to $Perf(Y)$ then $P_{\mathcal{E}} : D(Qcoh\, X) \longrightarrow D(Qcoh\, Y)$ is also fully faithful.

Kontsevich

Singular Lagrangian Submanifolds

$L \subset (X, \omega)$

(expand Fuk cat - cf. Coh is bigger than vec bund on holom submfd)



$U \supset L$

ε-nbhd = exact symp. mfd
= Liouville domain

$\exists$ vec field ξ

$L_\xi\, \omega = \omega$

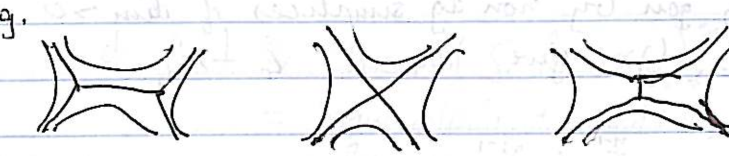
$\leftrightarrow \omega = d(i_\xi \omega)$

ξ outwards at ∂U

look at flow of ξ - everything that doesn't escape is our sing. Lag. (should have same homotopy type as U)

(if there is such a nbhd, our sing. Lag is 'good').

Choice of vector field = some sort of transformation, e.g.



diff. ξ

On L (with good stratification)

$\exists$ natural cosheaf in homotopy of finite type in dga/$\mathbb{Z}$ $\rightarrow \Phi_L$